

Exact Scalar-Tensor Cosmological Solutions via Noether Symmetry

J. A. Belinchón¹ • T. Harko² • M. K. Mak³

Abstract In this paper, we investigate the Noether symmetries of a generalized scalar-tensor, Brans-Dicke type cosmological model, in which we consider explicit scalar field dependent couplings to the Ricci scalar, and to the scalar field kinetic energy, respectively. We also include the scalar field self-interaction potential into the gravitational action. From the condition of the vanishing of the Lie derivative of the gravitational cosmological Lagrangian with respect to a given vector field we obtain three cosmological solutions describing the time evolution of a spatially flat Friedman-Robertson-Walker Universe filled with a scalar field. The cosmological properties of the solutions are investigated in detail, and it is shown that they can describe a large variety of cosmological evolutions, including models that experience a smooth transition from a decelerating to an accelerating phase.

Keywords scalar tensor cosmological models: exact solutions: Noether symmetry: cosmology: accelerating Universe.

1 Introduction

Despite the fact that general relativity (GR) has passed a variety of astrophysical and cosmological observational tests, from the weak post-Newtonian regime to

the strong field regime of neutron stars, or even black holes (Will 2006), and there are some progresses in quantum gravity (Rovelli 2004), presently it is generally accepted that there are many shortcomings of GR in both cosmology and quantum field theory.

In the past twenty years, in view of the various observational evidence obtained from the study of distant Type Ia Supernovae, it has been showed that the Universe has undergone a late time accelerated expansion (Riesz et al. 1998; Perlmutter et al. 1999; Knop et al. 2003; Amanullah et al. 2010). In order to explain this cosmological effect, a mysterious component, called dark energy, has been proposed for describing the late time evolution of the Universe. There are many models for dark energy, which try to explain the observational evidence of the recent acceleration of the Universe. For example, a canonical scalar field called quintessence was introduced as a candidate for dark energy (Tsujikawa 2013). Scalar fields were allowed to evolve in the Universe, while having couplings of order of unity to matter in the context of Chameleon cosmology (Khoury & Weltman 2004). The basic idea of the so called k -essence and dynamical attractors models is based on evolving scalar fields with non-linear kinetic energy terms in the action (Armendariz-Picon et al. 2001). An exotic fluid known as Chaplygin gas was introduced in order to unify dark matter and dark energy, with the equation of state $p = -A/\rho^\alpha$, where p is the pressure, ρ is the density, A is a positive constant and α is a constant (Bilic et al. 2002; Bento et al. 2002). The topics on phantom dark energy and cosmic doomsday were discussed in Caldwell et al. (2003). For reviews of dark energy models see (Sahni 2004; Copeland et al. 2006). Recently, from the observational point of view, the Cosmic Microwave Background (CMB) temperature anisotropies measured by the Wilkinson Microwave Anisotropy Probe and the Planck satellites were reported in (Spergel et al. 2003; Ade et al. 2013), con-

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firming that the total amount of baryonic matter in the Universe is very low. It is worth to note that the study of the baryonic acoustic oscillations have also confirmed the existence of the dark energy, and these results are also correlated with other cosmological observations (Percival et al. 2010).

Another approach to dark energy, related to the modified gravitational theories, is proposed in order to deal with the cosmic acceleration problem. In these models the geometric part of Einstein's gravitational field equation is modified. One of such modifications is the $f(R)$ gravity theory (Sotiriou & Faraoni 2010; De Felice & Tsujikawa 2010; Nojiri & Odintsov 2011; Capozziello & De Laurentis 2011). Note that Einstein was the first person to introduce the idea of teleparallel equivalent of GR for the sake of unifying the gravity and the electromagnetism (Einstein 1928a,b). In teleparallel gravity theory the gravitational field equations can be obtained by varying the action with respect to the vierbein fields (Hayashi & Shirafuji 1979).

It is generally recognized that scalar-tensor theory (STT) of gravity is the most straightforward generalization of standard GR theory. In 1961, the Brans-Dicke theory was formulated. In this formalism, an additional scalar field ϕ besides the metric tensor $g_{\mu\nu}$ and a dimensionless coupling constant ω were introduced in order to describe the gravitational interaction (Brans & Dicke 1961; Brans 1962; Brans & Dicke 1962; Dicke 1962; Dicke 1964). Brans-Dicke theory recovers the results of GR for large value of the coupling constant ω , that is for $\omega > 500$. Similar formalisms were developed earlier by Jordan (Jordan 1955, 1959). One can find the history and developments of STT in Fujii & Maeda (2003); Brans (2005); Goenner (2012); Sotiriou (2014). In 1974, the most general STT including four arbitrary functions of the scalar field and kinetic term was proposed by Horndeski Horndeski (1974). Horndeski's theory can be reduced to various modified gravity models by choosing the specific four functions. Thus this theory includes a wide class of modified gravity models, and hence we treat it as an effective theory of the generalized theories of gravity. A review of the effective field theory of the inflation/dark energy and the Horndeski's theory was presented in Tsujikawa (2015). In 2015, several pioneering works beyond Horndeski were presented in (Geyzes et al. 2015; Gleyzes et al. 2015; Kobayashi et al. 2015; De Felice et al. 2015).

Scalar-tensor theories are promising because they can provide an explanation of the inflationary behavior in cosmology. Note that the scalar field can be considered as an additional degree of freedom of the gravitational interaction. Scalar-tensor theories have been used to model dark energy, because the

scalar fields are good candidates for phantom and quintessence fields (Peeble & Rathra 2003; Harko et al. 2014; Mak & Harko 2002). Furthermore, it is natural to couple the scalar field to the curvature after compactification of higher dimensional theories of gravity such as Kaluza-Klein and string theory, thus offering the possibility of linking fundamental scalar fields with the nature of the dark energy. From observational point of view, the parameterized post-Newtonian (PPN) parameters γ and β for general STT was computed, and it was suggested that the PPN parameters γ and β given by the condition $|\gamma - 1| \sim |\beta - 1| \sim 10^{-6}$ may be detectable by a satellite that carries a clock with fractional frequency uncertainty $\frac{\Delta f}{f} \sim 10^{-16}$ in an eccentric orbit around the Earth (Schärer et al. 2014).

The weak field limit of STT of gravitation was discussed in view of conformal transformations in Stabile et al. (2013), and a new reconstruction method of STT based on the use of conformal transformations was proposed. This method allows the derivation of a set of interesting exact cosmological solutions in Brans-Dicke gravity, as well as in other extensions of GR (Vignolo et al. 2013). The phase space of Friedmann-Lemaître-Robertson-Walker models derived from STT in the presence of the non-minimal coupling $F(\phi) = \xi\phi^2$ and the effective potential $V(\phi) = \lambda\phi^n$ was studied in Carloni et al. (2008). Testing the feasibility of scalar-tensor gravity by scale dependent mass and coupling to matter was investigated in Mota et al. (2011). Recently, an isotropic model in the presence of variable gravitational and cosmological constants was studied in the context of scalar-tensor cosmology in Belinchón (2012). Generalized self-similar STT was investigated and some new exact self-similar solutions were obtained in Belinchón (2012), by using the Kantowski-Sach models.

A powerful mathematical method that allows to handle cosmological models coming from different fundamental gravity theories is represented by the Noether symmetry approach (Capozziello et al. 1996). With its use one can solve exactly the cosmological equations in several gravitational theories. The interest in such a mathematical approach becomes greater when the solutions found are physically interesting. In Capozziello et al. (1996) several cosmological solutions obtained by using Noether symmetries are presented. They are Friedmann, power law, pole-like and de-Sitter-like, and thus they can cover a large range of expected cosmological behaviors. From the observational point of view such Noether symmetrical solutions can be interpreted as the background on which one can compare theoretical models with the observational data from large-scale structure formation and Cosmic Mi-

crowave Background Radiation. These background solutions can be also used to formulate the theories of cosmological perturbations for a given model.

A complete Noether symmetry analysis in the framework of scalar-tensor cosmology was reported in Paliathanasis et al. (2014b) (see also (Terzis et al. 2014; Christodoulakis 2013) for other approaches). Scalar-tensor cosmology with $1/R$ curvature correction by Noether symmetry was discussed in Motavali et al. (2008). Recently, Noether symmetries of some homogeneous Universe models in curvature corrected scalar-tensor gravity was studied in Sharif & Waheed (2013). Using Noether symmetry approach, the solutions of the gravitational field equations in the context of scalar tensor teleparallel dark gravity was obtained in Kucukakca (2014). The evolution of teleparallel dark energy cosmological model with a fermionic field was studied in Kucukakca (2014). The evolution of a three-dimensional minisuperspace cosmological model via Noether symmetry was investigated in Vakili (2014). With the help of Noether symmetry, some cosmological analytical solutions for specific functional forms of coupling and potential functions were obtained in Paliathanasis et al. (2014a). The application of point symmetries in the recently proposed metric-Palatini hybrid gravity in order to select the $f(R^*)$ functional form and to find analytical solutions for the gravitational field equations and for the related Wheeler-DeWitt equation was considered in Borowiec et al. (2015), where R^* is the Palatini curvature scalar.

With the help of the standard mathematical procedure, the Lie group approach, and dynamical Noether symmetry techniques, we obtain several exact solutions of the gravitational field equations describing the time evolutions of a spatially flat Friedman-Robertson-Walker (FRW) Universe in the context of the scalar-tensor gravity. The obtained solutions can describe both accelerating and decelerating phases during the cosmological expansion of the Universe (Belinchón et al. 2015).

It is the purpose of the present paper to investigate the Noether symmetries of the cosmological dynamics of a modified STT gravitational model (Faraoni 2004; Bronnikov et al. 2002; Lee 2011), in which there is an explicit scalar field dependent coupling to the Ricci scalar, and to the scalar field kinetic energy, respectively are included in the gravitational action. In this model the scalar field self-interaction potential is also explicitly considered. From the condition of the vanishing of the Lie derivative of the gravitational cosmological Lagrangian with respect to a given vector field we obtain three cosmological solutions, describing the time

evolution of a spatially flat FRW Universe filled with a scalar field. The cosmological properties of these solutions are investigated in detail, and it is shown that they can describe a large variety of cosmological evolutions, including models that experience a smooth transition from a decelerating to an accelerating phase. Thus the presented solutions may offer a possible explanations of the observed cosmological dynamics, without the need of considering dark energy.

The present paper is organized as follows. We formulate the scalar-tensor model of gravity in Section 2. The Noether symmetries of this STT model are investigated, and two general analytical solutions with six particular solutions of the symmetry equation are presented in Section 3. The cosmological implications of three solutions are investigated in detail in Section 4. Finally, we discuss and conclude our results in Section 5.

2 Formalism of scalar-tensor gravity

In the present paper we consider a generalized scalar-tensor gravitational model, in which the Ricci scalar R , describing pure gravity, couples in a non-standard way with a scalar field ϕ . Moreover, we assume the second non-standard coupling between the scalar field and the kinetic energy term. Hence for this model the gravitational action takes the form (Faraoni 2004; Bronnikov et al. 2002; Lee 2011)

$$S = \int d^4x \sqrt{-g} \left[F(\phi) R + \frac{1}{2} Z(\phi) g^{\mu\nu} \phi_{;\mu} \phi_{;\nu} - V(\phi) \right], \quad (1)$$

where $F(\phi)$ is the generic function describing the coupling between the scalar field and geometry, $Z(\phi)$ is the arbitrary function coupling the scalar field to its kinetic energy, also distinguishing between various STT of gravity, and $V(\phi)$ is the scalar field potential.

In the following we investigate only the cosmological implications of this model. We consider the case of a spatially flat FRW Universe, described by the line element

$$ds^2 = dt^2 - a^2(t) (dx^2 + dy^2 + dz^2), \quad (2)$$

where $a(t)$ is the scale factor, which gives the information on the expansion of the universe. With the help of the line element (2), we obtain the Ricci scalar as

$$R = -6 \left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 \right], \quad (3)$$

where the overdot denotes the derivative with respect to the time t . The Lagrangian density on the configuration

space (a, ϕ) for the non-minimal coupled scalar-tensor cosmology described by action (1) takes the form

$$\begin{aligned}\mathcal{L} &= 6a\dot{a}^2 F(\phi) + 6a^2 \dot{a} \dot{\phi} F'(\phi) + a^3 \left[\frac{1}{2} Z(\phi) \dot{\phi}^2 - V(\phi) \right] \\ &= a^3 \left[6 \left(\frac{\dot{a}}{a} \right)^2 F(\phi) + 6 \frac{\dot{a}}{a} \dot{\phi} F'(\phi) + \frac{Z(\phi) \dot{\phi}^2}{2} - V(\phi) \right],\end{aligned}\quad (4)$$

where the prime denotes the derivative with respect to the scalar field ϕ . A first class of solutions of the model can be obtained by assuming that the Lagrangian \mathcal{L} is degenerate. Then, the Hessian \mathcal{H} defined mathematically as

$$\mathcal{H} = \begin{vmatrix} \frac{\partial^2 \mathcal{L}}{\partial \dot{a}^2} & \frac{\partial^2 \mathcal{L}}{\partial \dot{a} \partial \dot{\phi}} \\ \frac{\partial^2 \mathcal{L}}{\partial \dot{\phi} \partial \dot{a}} & \frac{\partial^2 \mathcal{L}}{\partial \dot{\phi}^2} \end{vmatrix}, \quad (5)$$

vanishes, $\mathcal{H} = 0$. By inserting Eq. (4) into Eq. (5), and using the condition for degeneracy $\mathcal{H} = 0$, the latter condition gives

$$\begin{vmatrix} 12aF & 6a^2 \frac{dF}{d\phi} \\ 6a^2 \frac{dF}{d\phi} & a^3 Z \end{vmatrix} = 12a^4 \left[Z(\phi)F(\phi) - 3 \left(\frac{dF(\phi)}{d\phi} \right)^2 \right] = 0.$$

Immediately Eq. (6) can be integrated to yield the integral equation

$$2\sqrt{F(\phi)} + \frac{\sqrt{3}}{3} \int \sqrt{Z(\phi)} d\phi \pm C = 0, \quad (7)$$

where C is an arbitrary constant of integration. By choosing some specific forms for one of the arbitrary functions $F(\phi)$ or $Z(\phi)$ we can fix the coupling functions of the model.

For example, if $F(\phi) = \mu^2 \phi^{2s}/4$, with μ and s arbitrary constants, from Eq. (6) then we obtain $Z(\phi) = 3\mu^2 s^2 \phi^{2s-2}$. By assuming for $Z(\phi)$ the functional form $Z(\phi) = 3m_1^2 Z_1^2 \phi^{2(m_1-1)}$, where m_1 and Z_1 are arbitrary constants, we obtain for $F(\phi)$ the expression $F(\phi) = (C - Z_1 \phi^{m_1})^2/4$, where we have adopted the minus sign for the constant C in Eq. (7).

If we fix both $F(\phi)$ and $Z(\phi)$ as having a polynomial form in ϕ , we obtain an algebraic equation which has the solution given by $\phi = \text{constant}$. In this case the physical model described by the degenerate Lagrangian (4) reduces, after a rescaling of the parameters, to standard GR, with the potential $V(\phi)$ generating a cosmological constant. Note that the conditions $\frac{\partial \mathcal{L}}{\partial t} = 0$ and $\mathcal{H} \neq 0$ hold for the time-independent, non-degenerate Lagrangians $\mathcal{L} = \mathcal{L}(q^i, \dot{q}^j)$ Capozziello & Lambiasi (2000).

Next, in order to obtain some cosmological solutions describing the time evolution of the FRW Universe in the context of the cosmological model described by the action (1), we will first investigate in the following Section its Noether symmetries.

3 Noether symmetries

We define the configuration space Q of the cosmological model as $Q \equiv (a, \phi)$, with the tangent space TQ given by $TQ \equiv (a, \dot{a}, \phi, \dot{\phi})$. Using the infinitesimal generator of the Noether symmetry, the lift vector \mathbf{X} can now be written as

$$\mathbf{X} = \alpha(a, \phi) \frac{\partial}{\partial a} + \beta(a, \phi) \frac{\partial}{\partial \phi} + \frac{d\alpha}{dt} \frac{\partial}{\partial \dot{a}} + \frac{d\beta}{dt} \frac{\partial}{\partial \dot{\phi}}, \quad (8)$$

where α, β are functions of the scale factor a and the scalar field ϕ . The existence of the Noether symmetry implies the existence of a vector field \mathbf{X} , so that we have the relation (Capozziello & Lambiasi 2000)

$$L_{\mathbf{X}} \mathcal{L} = \mathbf{X} \mathcal{L} = 0, \quad (9)$$

where $L_{\mathbf{X}}$ stands for the Lie derivative with respect to the vector field \mathbf{X} . The existence of the symmetry \mathbf{X} (6) provides us a constant of motion, through the Noether theorem. The constant of motion Σ , defined by the inner derivative, takes the form

$$\Sigma = i_{\mathbf{X}} \theta_{\mathcal{L}}, \quad (10)$$

where the Cartan one form is defined by

$$\theta_{\mathcal{L}} = \frac{\partial \mathcal{L}}{\partial \dot{a}} da + \frac{\partial \mathcal{L}}{\partial \dot{\phi}} d\phi. \quad (11)$$

Therefore, with the help of Eqs. (4) and (9), we obtain the following system of partial differential equations

$$F(\alpha + 2a\partial_a \alpha) + aF'(\beta + a\partial_a \beta) = 0, \quad (12)$$

$$3Z\alpha + Z'a\beta + 12F'\partial_\phi \alpha + 2Za\partial_\phi \beta = 0, \quad (13)$$

$$a\beta F'' + (2\alpha + a\partial_a \alpha + a\partial_\phi \beta) F' + 2F\partial_\phi \alpha + \frac{1}{6} Z a^2 \partial_a \beta = 0, \quad (14)$$

$$3\alpha V + a\beta V' = 0. \quad (15)$$

In the following we shall obtain two classes of solutions of the system (12)-(15), and we shall investigate their cosmological properties.

3.1 Solution 1

(21)

It is easy to show by direct computations that Eqs. (12)-(15) have the following solution,

$$\begin{aligned} F(\phi) &= K\phi^n, Z(\phi) = Z_0\phi^{n-2}, V(\phi) = V_0\phi^n, \\ \alpha(a) &= c_1 a, \beta(\phi) = -\frac{3c_1}{n}\phi, \end{aligned} \quad (16)$$

where K, Z_0, V_0, n and c_1 are the arbitrary constants.

3.2 Solution 2

After tedious calculations, we integrate Eqs. (12)-(15) to obtain a second class of solutions as

$$F(\phi) = K\phi^n, V(\phi) = V_0\phi^{\frac{3c_1 n}{2c_1+1}}, \quad (17)$$

$$Z(\phi) = \frac{3Kn^2 \left[\frac{4c_2 c_1 (c_1+1)}{(2c_1+1)^2} \phi^{\frac{3c_1 n}{2c_1+1}} - c_3 \phi^n \right]}{\left(c_2 \phi^{\frac{(c_1-1)n}{2c_1+1}} - c_3 \right) \phi^2}, \quad (18)$$

$$\begin{aligned} \alpha(a, \phi) &= \frac{a^{c_1}}{n(1-c_1)} \times \\ &\sqrt{2n(2c_1+1)(1-c_1) \left[c_2 \phi^{\frac{2(c_1^2-1)n}{2c_1+1}} - c_3 \phi^{n(c_1-1)} \right]}, \end{aligned} \quad (19)$$

$$\begin{aligned} \beta(a, \phi) &= \frac{1}{c_1} \sqrt{\frac{2}{1-c_1} \left(\frac{2c_1+1}{n} \right)^3} \times \\ &\frac{a^{c_1-1} \phi \left(c_3 - c_2 \phi^{\frac{(c_1-1)n}{2c_1+1}} \right)}{\sqrt{c_2 \phi^{\frac{2c_1(1-c_1)n}{2c_1+1}} - c_3 \phi^{n(1-c_1)}}}, \end{aligned} \quad (20)$$

where the arbitrary constant c_1 satisfies the relations $c_1 \neq 1$ and $c_1 \neq -1/2$, and c_2 and c_3 are arbitrary constants of integration.

In order to investigate in detail the cosmological implications of the present solutions, in the following we shall concentrate on two particular solutions of Eqs. (18)-(20), corresponding to $c_2 = 0$ and $c_3 = 0$, respectively.

3.2.1 Specific solution $c_2 = 0$ and $n \neq 0$

Assume that the arbitrary constant c_2 vanishes, then from Eqs. (18)-(20),

$$F(\phi) = K\phi^n, V(\phi) = V_0\phi^{\frac{3c_1 n}{2c_1+1}}, Z(\phi) = 3Kn^2\phi^{n-2},$$

$$\alpha(a, \phi) = \frac{\sqrt{2n(2c_1+1)(c_1-1)c_3}}{n(1-c_1)} a^{c_1} \phi^{n(c_1-1)/2}, \quad (22)$$

$$\beta(a, \phi) = \frac{1}{c_1} \sqrt{\frac{2c_3}{c_1-1} \left(\frac{2c_1+1}{n} \right)^3} a^{c_1-1} \phi^{1-\frac{n(1-c_1)}{2}}. \quad (23)$$

Now, we shall present two specific solutions of Eqs. (21)-(23) for $n = 1, 2$ respectively.

For $n = 2$, from Eqs. (21)-(23), we obtain the following solution for the generalized STT model,

$$F(\phi) = K\phi^2, V(\phi) = V_0\phi^{\frac{6c_1}{2c_1+1}}, Z(\phi) = 12K, \quad (24)$$

$$\alpha(a, \phi) = \frac{\sqrt{(2c_1+1)(c_1-1)c_3}}{(1-c_1)} a^{c_1} \phi^{c_1-1}, \quad (25)$$

$$\beta(a, \phi) = \frac{1}{2c_1} \sqrt{\frac{c_3(2c_1+1)^3}{c_1-1}} a^{c_1-1} \phi^{c_1}. \quad (26)$$

For $n = 1$, we obtain the Jordan-Brans-Dicke type solutions,

$$F(\phi) = K\phi, V(\phi) = V_0\phi^{\frac{3c_1}{2c_1+1}}, Z(\phi) = 3K\phi^{-1}, \quad (27)$$

$$\alpha(a, \phi) = \frac{\sqrt{2(2c_1+1)(c_1-1)c_3}}{(1-c_1)} a^{c_1} \phi^{(c_1-1)/2}, \quad (28)$$

$$\beta(a, \phi) = \frac{1}{c_1} \sqrt{\frac{2c_3(2c_1+1)^3}{c_1-1}} a^{c_1-1} \phi^{\frac{1+c_1}{2}}. \quad (29)$$

3.2.2 Specific solution $c_3 = 0$ and $n \neq 0$

Assume that the arbitrary constant c_3 vanishes, then from Eqs. (18)-(20), we obtain the solutions

$$\begin{aligned} F(\phi) &= K\phi^n, V(\phi) = V_0\phi^{\frac{3c_1 n}{2c_1+1}}, \\ Z(\phi) &= \frac{12Kn^2 c_1 (c_1+1)}{(2c_1+1)^2} \phi^{n-2}, \end{aligned} \quad (30)$$

$$\alpha(a, \phi) = \sqrt{\frac{2(2c_1+1)c_2}{n(1-c_1)}} a^{c_1} \phi^{\frac{n(c_1^2-1)}{2c_1+1}}, \quad (31)$$

$$\beta(a, \phi) = -\frac{1}{c_1} \sqrt{\frac{2c_2}{1-c_1} \left(\frac{2c_1+1}{n} \right)^3} a^{c_1-1} \phi^{\frac{n(c_1^2-1)}{2c_1+1}+1}. \quad (32)$$

Assume that $n = 2$ and $c_1 = -2$, then from Eqs. (30)-(32), we obtain the results

$$\begin{aligned} F(\phi) &= K\phi^2, V(\phi) = V_0\phi^4, Z(\phi) = \frac{32K}{3}, \\ \alpha(a, \phi) &= \frac{\sqrt{-c_2}}{(a\phi)^2}, \beta(a, \phi) = \frac{3\sqrt{-c_2}}{4a^3\phi}, c_2 < 0, \end{aligned} \quad (33)$$

which are similar to the solutions given by Capozziello & de Ritis (1994). Furthermore for $n = 2$ and $c_1 = 1/4$, we obtain the solutions

$$\begin{aligned} F(\phi) &= K\phi^2, V(\phi) = V_0\phi, Z(\phi) = \frac{20K}{3}, \\ \alpha(a, \phi) &= \sqrt{2c_2} (a\phi^{-5})^{1/4}, \\ \beta(a, \phi) &= -\frac{3\sqrt{2c_2}}{(a^3\phi)^{1/4}}, c_2 > 0. \end{aligned} \quad (34)$$

Hence by using the requirement of the existence of a Noether symmetry, and with the help of the vector field \mathbf{X} given by Eq. (8), of the Lagrangian density (4), we have obtained some solutions of the system (12)-(15), in the FRW geometry (2). In the following Sections we shall investigate the cosmological solutions in the framework of the general STT corresponding to these Noether symmetries.

4 Exact cosmological solutions

4.1 Cosmological solution 1

Using the Noether symmetries presented in Section III, we have found a set of solutions of the symmetry equations, given by Eqs. (16). Now in order to find the explicit forms of the scale factor $a(t)$ and of the scalar field $\phi(t)$, we introduce two arbitrary functions z and w defined as

$$z = z(a, \phi), \quad w = w(a, \phi), \quad (35)$$

respectively. The transformed Lagrangian is cyclic in one of the new variables. Using the relations $i_{\mathbf{X}}(dz) = 1$ and $i_{\mathbf{X}}(dw) = 0$, we obtain the differential equations

$$\alpha\partial_a z + \beta\partial_\phi z = 1, \quad (36)$$

$$\alpha\partial_a w + \beta\partial_\phi w = 0, \quad (37)$$

respectively. Now in Eqs. (16) without loss of generality, we set $c_1 = 1$ in the following. By substituting Eqs. (16) into Eqs. (36) and (37), the latter equations can respectively be integrated to yield

$$z(a, \phi) = \ln a + \phi a^{3/n}, \quad w(a, \phi) = \phi a^{3/n}, \quad (38)$$

$$a(w, z) = e^{z-w}, \quad \phi(w, z) = we^{\frac{3}{n}(w-z)}. \quad (39)$$

With the help of Eqs. (4), (16), (38), and (39), and by setting $n \neq 0$, $Z_0 = 1$, and $8n^2K = 3$, the Lagrangian (4) takes the form

$$\mathcal{L} = \frac{3}{4n} \dot{w} w^{n-1} (\dot{w} - \dot{z}) + \frac{1}{2} w^{n-2} \dot{w}^2 - V_0 w^n. \quad (40)$$

Next the Euler-Lagrange equations are given by

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{z}} \right) = \frac{\partial \mathcal{L}}{\partial z}, \quad (41)$$

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{w}} \right) = \frac{\partial \mathcal{L}}{\partial w}, \quad (42)$$

respectively. By inserting Eq. (40) into Eqs. (41) and (42), the latter equations yield

$$\frac{d}{dt} (w^{n-1} \dot{w}) = 0, \quad (43)$$

$$\begin{aligned} \ddot{z} &= 2 \left(1 + \frac{2n}{3w} \right) \ddot{w} + \frac{2}{3} n(n-2) \left(\frac{\dot{w}}{w} \right)^2 + \\ &\quad (n-1) \frac{\dot{w}^2}{w} + \frac{4}{3} V_0 n^2, \end{aligned} \quad (44)$$

respectively. Eq. (43) can be easily integrated to give

$$w(t) = [n(k_1 t + k_2)]^{1/n}, \quad (45)$$

where k_1 and k_2 are two arbitrary constants of integration. By inserting Eq. (45) into Eq. (44), the latter can be integrated to give

$$z(t) = [n(k_1 t + k_2)]^{1/n} + f(t) + \frac{2}{3} \ln |k_1 t + k_2| + k_4, \quad (46)$$

where for simplicity we have denoted the arbitrary function $f(t)$ as

$$f(t) = t \left[\frac{2}{3} n^2 V_0 \left(t - \frac{2k_2}{k_1} \right) + k_3 \right], \quad (47)$$

where k_3 and k_4 are arbitrary constants of integration. Now by inserting Eqs. (45) and (46) into Eq. (39), this allows us to obtain the scale factor a , the Hubble function $H = \dot{a}/a$ and the scalar field ϕ as

$$a(t) = e^{k_4} (k_1 t + k_2)^{2/3} e^{f(t)}, \quad (48)$$

$$H(t) = \frac{1}{3} \left[\frac{4n^2 V_0 (k_1 t - k_2)}{k_1} + \frac{2k_1}{k_1 t + k_2} + 3k_3 \right], \quad (49)$$

$$\phi^n(t) = \frac{n}{e^{3k_4} (k_1 t + k_2) e^{3f(t)}}, \quad (50)$$

respectively.

By inserting Eq. (50) into the relation given by $V(\phi) = V_0\phi^n$, thus we easily obtain the potential

$$V(t) = \frac{nV_0}{e^{3k_4}} \frac{1}{(k_1t + k_2)e^{3f(t)}}. \quad (51)$$

An important observational quantity is the deceleration parameter q defined as

$$q(t) = \frac{d}{dt} \left(\frac{1}{H} \right) - 1. \quad (52)$$

The sign of the deceleration parameter indicates whether the cosmological model accelerates, or not. The positive sign of q corresponds to standard decelerating models whereas the negative sign indicates acceleration. Now by substituting Eq. (48) into Eq. (52), the latter becomes

$$q(t) = \frac{6 \left[\frac{k_1^2}{(k_1t + k_2)^2} - 2n^2V_0 \right]}{\left[4n^2V_0 \left(t - \frac{k_2}{k_1} \right) + \frac{2k_1}{(k_1t + k_2)} + 3k_3 \right]^2} - 1. \quad (53)$$

It is important to note that for $t \rightarrow 0$ both the scalar field and its potential are finite if $k_2 \neq 0$. In order to determine the values of the integration constants k_i , $i = 1, \dots, 4$ we proceed as follows. We assume first that at $t = 0$, $a(0) = a_0$, a condition which gives, with the use of Eq. (48) $a_0 = e^{k_4}k_2^{2/3}$, or $k_2 = a_0^{3/2}$. Hence we can take $k_4 = 0$ without any loss of generality. The condition $H(0) = H_0$ gives

$$H_0 = -\frac{4n^2V_0k_2}{3k_1} + \frac{2k_1}{3k_2} + k_3. \quad (54)$$

Again, without any loss of generality we can fix the constant k_3 as $k_3 = 0$. Then from the above equation we obtain for the ratio of k_1/k_2 given by

$$\frac{k_1}{k_2} = \frac{3H_0 + \sqrt{9H_0^2 + 32n^2V_0}}{4} = \frac{3H_0}{4} \left(1 + \sqrt{1 + \chi} \right), \quad (55)$$

where we have denoted $\chi = 32n^2V_0/9H_0^2$. Therefore the cosmological parameters corresponding to the first Noether symmetry solution can be written in the form

$$a(t) = a_0 \left[\frac{3}{4} \left(1 + \sqrt{1 + \chi} \right) H_0 t + 1 \right]^{2/3} \times e^{\frac{\chi}{16} H_0 t \left(3H_0 t - \frac{8}{1 + \sqrt{1 + \chi}} \right)}, \quad (56)$$

$$H(t) = H_0 \left[\frac{\chi}{8} \left(3H_0 t - \frac{4}{1 + \sqrt{1 + \chi}} \right) + \frac{2(1 + \sqrt{1 + \chi})}{3(1 + \sqrt{1 + \chi}) H_0 t + 4} \right], \quad (57)$$

$$\phi^n(t) = \frac{4n}{a_0^{3/2}} \frac{e^{\frac{3\chi}{16} H_0 t \left(\frac{8}{1 + \sqrt{1 + \chi}} - 3H_0 t \right)}}{\left[3(1 + \sqrt{1 + \chi}) H_0 t + 4 \right]}, \quad (58)$$

$$V(t) = \frac{4nV_0}{a_0^{3/2}} \frac{e^{\frac{3\chi}{16} H_0 t \left(\frac{8}{1 + \sqrt{1 + \chi}} - 3H_0 t \right)}}{\left[3(1 + \sqrt{1 + \chi}) H_0 t + 4 \right]}, \quad (59)$$

$$q(t) = -24 \left(\sqrt{\chi + 1} + 1 \right)^2 \left[9\chi \left(\chi + 2\sqrt{\chi + 1} + 2 \right) \times \left(H_0 t \right)^2 + 32 \left(\sqrt{\chi + 1} + 1 \right) \right]^{-2} \left\{ 3\chi \left[3H_0 t \left(\chi + 2\sqrt{\chi + 1} + 2 \right) + 8 \left(\sqrt{\chi + 1} + 1 \right) \right] H_0 t - 32 \left(\sqrt{\chi + 1} + 1 \right) \right\} - 1. \quad (60)$$

The time variations of the scale factor, of the Hubble function, of the scalar field potential and of the deceleration parameter for the first Noether symmetry solution of the modified STT cosmological model are represented in Figs. 1-4 respectively

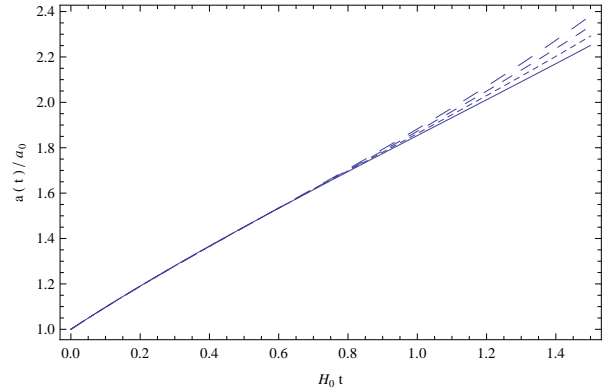


Fig. 1 Time variation of the scale factor $a(t)$ in the first Noether symmetry solution of the generalized STT cosmological model for different values of the parameter χ : $\chi = 0.15$ (solid curve), $\chi = 0.25$ (dotted curve), $\chi = 0.35$ (dashed curve), and $\chi = 0.45$ (long dashed curve), respectively.

As one can see from Fig. 1, the Universe described by the first Noether symmetry solution of the generalized STT experiences an expansionary phase, with the scale factor a monotonically increasing function of time. The Hubble function, presented in Fig. 2, is a monotonically decreasing function of time in the early stages of cosmological evolution, but, after reaching a minimum value, for large values of the parameter χ it becomes a

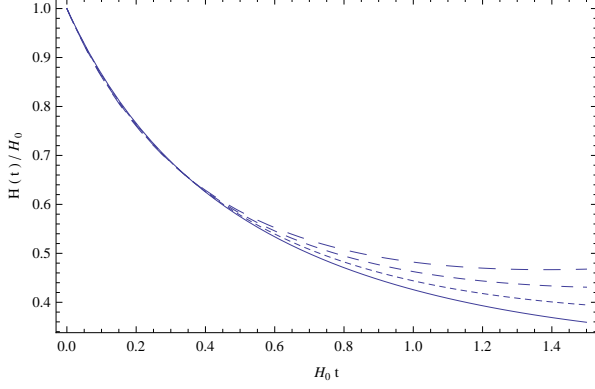


Fig. 2 Time variation of the Hubble function $H(t)$ in the first Noether symmetry solution of the generalized STT cosmological model for different values of the parameter χ : $\chi = 0.15$ (solid curve), $\chi = 0.25$ (dotted curve), $\chi = 0.35$ (dashed curve), and $\chi = 0.45$ (long dashed curve), respectively.

monotonically increasing function of time, thus indicating the possibility of the reversal of the expansion to a collapsing phase. The scalar field potential, depicted in Fig. 3, is a monotonically decreasing function of time, for all considered values of the parameter χ . The deceleration parameter q , plotted in Fig. 4, shows that in the present model the Universe starts its evolution in a decelerating phase, with $q > 0$. However, in all cases the Universe enters in an accelerating phase, with $q < 0$, with the time spent in the decelerating phase depending on the numerical value of χ . For large values of χ , in the large time limit, the Universe ends its evolution with $q = -1.5$, thus indicating that the present model allows super de Sitter accelerating cosmological expansions.

4.1.1 The constant of motion

Since the constant of motion Σ plays an important role in the study of Noether symmetries, in the following we shall obtain the explicit form for Σ . Using Eqs. (10) and (11), we find for the constant of motion Σ given by

$$\Sigma = \alpha \frac{\partial \mathcal{L}}{\partial \dot{a}} + \beta \frac{\partial \mathcal{L}}{\partial \dot{\phi}}. \quad (61)$$

By substituting Eqs. (4) and (16) into the Eqs. (61), the latter takes the form

$$\Sigma = -6K a^2 \dot{a} \phi^n + 3 \left(2nK - \frac{1}{n} \right) a^3 \dot{\phi} \phi^{n-1}. \quad (62)$$

We rewrite Eq. (62) as a Bernoulli differential equation for the scalar field ϕ^n in the form

$$\frac{d\phi^n}{dt} = \frac{2Kn^2}{2Kn^2 - 1} \frac{d \ln a}{dt} \phi^n + \frac{n^2 \Sigma}{3(2Kn^2 - 1)a^3}. \quad (63)$$

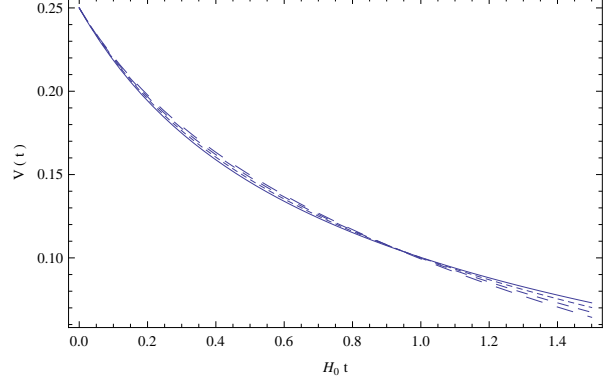


Fig. 3 Time variation of the scalar field potential $V(t)$ in the first Noether symmetry solution of the generalized STT cosmological model for different values of the parameter χ : $\chi = 0.15$ (solid curve), $\chi = 0.25$ (dotted curve), $\chi = 0.35$ (dashed curve), and $\chi = 0.45$ (long dashed curve), respectively. For the constant V_0 we have adopted the value $V_0 = a_0^{3/2}/4n$.

The general solution of Eq. (63) is given by

$$\phi^n(a, t) = a^{\frac{2Kn^2}{2Kn^2 - 1}} \left[\phi_0 + \frac{n^2 \Sigma}{3(2Kn^2 - 1)} \int a^{\frac{8Kn^2 - 3}{1 - 2Kn^2}} dt \right], \quad (64)$$

where ϕ_0 is the arbitrary constant of integration. Since we have already used $8Kn^2 = 3$ or equivalently $n_{\pm} = \pm \sqrt{\frac{3}{8K}}$, thus Eq. (64) has a simple form

$$\phi^n(a, t) = a^{-3} \left(\phi_0 - \frac{4n^2 \Sigma}{3} t \right). \quad (65)$$

Furthermore, if the constant of motion Σ vanishes, then Eq. (65) will take a simpler form

$$\phi^n(a) = \phi_0 a^{-3}. \quad (66)$$

The result given by Eq. (66) can be reobtained from the differential equation $\frac{da}{\alpha} = \frac{d\phi}{\beta}$, where $\alpha(a) = a$, $\beta(\phi) = -\frac{3}{n}\phi$.

4.2 Cosmological solution 2

Using the Noether symmetries presented in section III, we have found the solutions given by Eqs. (30)-(32). Here we recall these solutions again

$$F(\phi) = K\phi^n, V(\phi) = \lambda\phi^{3c_1b_3}, Z(\phi) = b_0\phi^{n-2}, \quad (67)$$

$$\alpha(a, \phi) = b_1 a^{c_1} \phi^{b_4}, \quad (68)$$

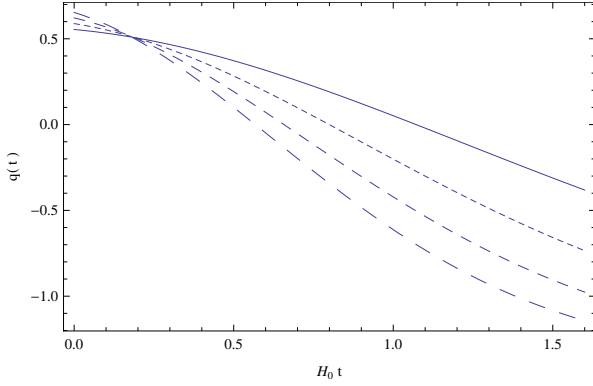


Fig. 4 Time variation of the deceleration parameter $q(t)$ in the first Noether symmetry solution of the generalized STT cosmological model for different values of the parameter χ : $\chi = 0.15$ (solid curve), $\chi = 0.25$ (dotted curve), $\chi = 0.35$ (dashed curve), and $\chi = 0.45$ (long dashed curve), respectively.

$$\beta(a, \phi) = b_2 a^{c_1-1} \phi^{1+b_4}, \quad (69)$$

where for simplicity we have denoted

$$b_0 = \frac{12K n^2 c_1 (1+c_1)}{(1+2c_1)^2}, b_1 = \sqrt{\frac{2(1+2c_1)c_2}{n(1-c_1)}},$$

$$b_2 = -\frac{1}{c_1} \sqrt{\frac{2c_2}{1-c_1} \left(\frac{1+2c_1}{n}\right)^3}, b_3 = \frac{n}{1+2c_1},$$

$$b_4 = b_3 (c_1^2 - 1),$$

and $V_0 = \lambda$. By inserting Eqs. (68) and (69) into the differential equation $da/\alpha = d\phi/\beta$, thus integrating the latter equation yields the invariant solution induced by the Noether symmetry

$$\phi(a) = \phi_1 a^{-\frac{1+2c_1}{nc_1}}, \quad (70)$$

where ϕ_1 is an arbitrary constant of integration. By inserting Eqs. (68) and (69) into Eqs. (37) and (36), we obtain the solutions of Eqs. (37) and (36) as given by

$$w(a, \phi) = \phi a^{\frac{1+2c_1}{nc_1}}, \quad (71)$$

$$z(a, \phi) = \gamma^{-1} a^{1-c_1} \phi^{\frac{n(1-c_1^2)}{1+2c_1}}. \quad (72)$$

With the help of Eqs. (71) and (72), we obtain the scale factor a and the scalar field ϕ given by

$$a(w, z) = \left[\gamma z w^{\frac{n(c_1^2-1)}{1+2c_1}} \right]^{\frac{c_1}{c_1-1}}, \quad (73)$$

$$\phi(w, z) = (\gamma z)^{\frac{1+2c_1}{n(1-c_1)}} w^{-c_1}, \quad (74)$$

respectively, where for simplicity we have denoted an arbitrary constant γ as

$$\gamma = -\frac{1}{c_1} \sqrt{\frac{2c_2(1-c_1)(1+2c_1)}{n}}. \quad (75)$$

In order to have real functions $a(w, z)$ and $\phi(w, z)$ as given by Eqs. (73) and (74), thus we impose the conditions $c_1 < 0$, and

$$\frac{2c_2(1-c_1)(1+2c_1)}{n} > 0, \quad (76)$$

respectively. By inserting Eqs. (67), (73) and (74) into the Lagrangian (4), we obtain the Lagrangian

$$\mathcal{L} = -\lambda w^{\frac{3nc_1}{1+2c_1}} - s_0 \dot{w} \dot{z} w^{\frac{nc_1(2+c_1)}{1+2c_1}-1}, \quad (77)$$

expressed in the new variables w and z , where we have denoted the arbitrary constant s_0 as

$$s_0 = 6K \sqrt{\frac{2nc_2}{(1-c_1)(1+2c_1)}}, \quad (78)$$

and $c_1 \neq -1/2$. In the following we shall present two cosmological solutions describing the dynamics of the FRW Universe for the conditions $c_1 \neq -2$ and $c_1 = -2$ respectively.

4.2.1 Specific case $c_1 \neq -2$

For the case $c_1 \neq -2$, again by inserting Eq. (77) into the Euler-Lagrange Eqs. (41) and (42), we obtain

$$\frac{d}{dt} \left(\dot{w} w^{\frac{nc_1(2+c_1)}{1+2c_1}-1} \right) = 0, \quad (79)$$

and

$$\begin{aligned} \ddot{z} &= \frac{\lambda}{s_0} \left(\frac{3nc_1}{1+2c_1} \right) w^{\frac{nc_1(1-c_1)}{1+2c_1}} = \\ &= \frac{\lambda}{s_0} \left(\frac{3nc_1}{1+2c_1} \right) (w_0 s_1 t + s_2)^{\frac{1-c_1}{2+c_1}}, \end{aligned} \quad (80)$$

respectively. By integrating Eqs. (79) and (80) yields

$$w(t) = (w_0 s_1 t + s_2)^{\frac{1+2c_1}{nc_1(2+c_1)}}, \quad (81)$$

$$\begin{aligned} z(t) &= \frac{\lambda}{s_0} \left(\frac{3nc_1}{1+2c_1} \right) \int^t \int^{t_2} (w_0 s_1 t_1 + s_2)^{\frac{1-c_1}{2+c_1}} \times \\ &\quad dt_1 dt_2 + s_3 t + s_4, \frac{1-c_1}{2+c_1} \neq -1, \end{aligned} \quad (82)$$

where we have used Eq. (81) for obtaining Eq. (82), and we have denoted s_i , $i = 1, 2, 3, 4$ as the arbitrary constants of integration. We have also denoted the arbitrary constant w_0 as $w_0 = \frac{nc_1(2+c_1)}{1+2c_1}$. By inserting Eq. (81) into Eq. (82), then the latter takes the form

$$z(t) = \frac{\lambda z_0}{s_0} (w_0 s_1 t + s_2)^{\frac{5+c_1}{2+c_1}} + s_3 t + s_4, \quad (83)$$

where we have denoted the arbitrary constant z_0 as $z_0 = \frac{1+2c_1}{(5+c_1)nc_1s_1^2}$. By substituting Eqs. (81) and (83) into Eq. (73), then we obtain the scale factor $a(t)$ as

$$a(t) = \left\{ \gamma \left[\frac{\lambda z_0}{s_0} (w_0 s_1 t + s_2)^{\frac{5+c_1}{2+c_1}} + s_3 t + s_4 \right] \right\}^{\frac{c_1}{c_1-1}} \times (w_0 s_1 t + s_2)^{\frac{1+c_1}{2+c_1}}. \quad (84)$$

In order to simplify the obtained cosmological expressions we rescale the time coordinate and the scale factor as $t \rightarrow t_0 + t/w_0 s_1$ and $a(t) \rightarrow a(t)\gamma^{c_1/(c_1-1)}$, respectively. We fix the constants t_0 , s_1 , s_2 , s_3 and s_4 through the relations $w_0 s_1 t_0 + s_2 = 0$, and $s_3 t_0 + s_4 = 0$, respectively. Then, by denoting

$$\frac{\lambda z_0}{s_0} = \frac{\sigma_0 s_3}{w_0 s_1} = \frac{s_4}{s_2} = \sigma, \quad (85)$$

and $m = 3/(2 + c_1)$ respectively, we obtain first

$$a(t) = (\sigma_0 t^m + \sigma)^{\frac{3-2m}{3(1-m)}} t^{\frac{6+m(m-6)}{3(1-m)}} = a_0 \left[\left(\frac{t}{t_0} \right)^m + 1 \right]^{\frac{3-2m}{3(1-m)}} \left(\frac{t}{t_0} \right)^{\frac{6+m(m-6)}{3(1-m)}}, \quad (86)$$

where $a_0 = \sigma^{\frac{3-2m}{3(1-m)}} t_0^{\frac{6+m(m-6)}{3(1-m)}}$ and $t_0 = (\sigma/\sigma_0)^{1/m}$.

For the Hubble function $H(t)$, and for the deceleration parameter $q(t)$ we obtain

$$H(t) = \frac{-m^2 + [m(m+3) - 6] \left(\frac{t}{t_0} \right)^m + 6m - 6}{3(m-1)t \left[\left(\frac{t}{t_0} \right)^m + 1 \right]}. \quad (87)$$

and

$$q(t) = 3 \left\{ m^2 + [6 - m(m+3)] \left(\frac{t}{t_0} \right)^m - 6m + 6 \right\}^{-2} \times \left\{ -m^3 + 7m^2 + \left[m(m((5-2m)m+6) - 21) + 12 \right] \left(\frac{t}{t_0} \right)^m + (m-1)[m(m+3) - 6] \left(\frac{t}{t_0} \right)^{2m} - 12m + 6 \right\} - 1,$$

respectively.

By substituting Eqs. (81) and (83) into Eq. (74), then the scalar field $\phi(t)$ is given by

$$\phi(t) = \phi_0 \left[\left(\frac{t}{t_0} \right)^m + 1 \right]^{\frac{2-m}{(m-1)n}} \left(\frac{t}{t_0} \right)^{\frac{(m-2)^2}{(m-1)n}}, \quad (89)$$

where we have rescaled the scalar field as

$$\phi(t) \rightarrow \phi(t) \gamma^{\frac{1+2c_1}{n(1-c_1)}}, \quad (90)$$

and ϕ_0 is given by

$$\phi_0 = \sigma^{\frac{2-m}{(m-1)n}} t_0^{\frac{(m-2)^2}{(m-1)n}}. \quad (91)$$

With the help of Eq. (89), the potential $V(t)$ can be easily obtained from the relation $V(\phi) = \lambda \phi^{3c_1 b_3}$ as

$$V(\phi) = \lambda \phi^{\frac{3nc_1}{1+2c_1}} = \lambda \phi^{\frac{n(2m-3)}{m-2}}. \quad (92)$$

The plots for the scale factor a , Hubble function H , deceleration parameter q and the potential V are presented in Figs. 5-8, respectively.

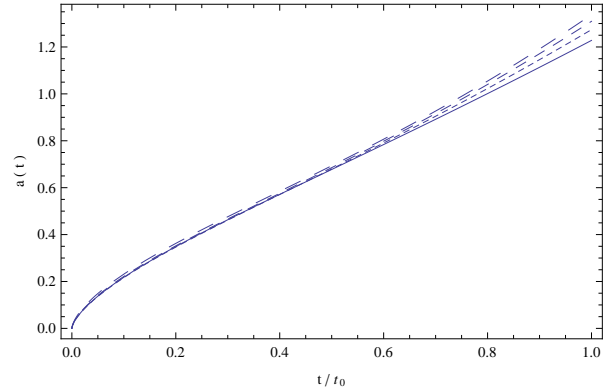


Fig. 5 Time variation of the scale factor $a(t)$ in the second Noether symmetry solution of the generalized STT cosmological model for different values of the parameter m : $m = 1.95$ (solid curve), $m = 2.05$ (dotted curve), $m = 2.20$ (dashed curve), and $m = 2.355$ (long dashed curve), respectively. The value of the constant a_0 has been fixed as $a_0 = 1$.

The scale factor of this cosmological model, represented in Fig. 5, shows that for the adopted values of the parameter m the universes are in a state of expansion, with a monotonically increasing in time. The Hubble function, depicted in Fig. 6, is a monotonically decreasing function of time, tending, in the large time limit, to a constant value. The scalar field potential, depicted in Fig. 7, shows a complex dynamical behavior. The potential is singular for $t \rightarrow 0$. Then it decreases monotonically in time, reaches a minimum value, and begins again to increase. The deceleration parameter, shown

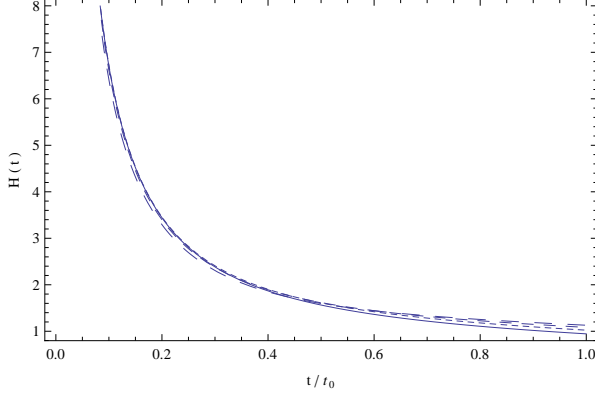


Fig. 6 Time variation of the Hubble function $H(t)$ in the second Noether symmetry solution of the generalized STT cosmological model for different values of the parameter m : $m = 1.95$ (solid curve), $m = 2.05$ (dotted curve), $m = 2.20$ (dashed curve), and $m = 2.355$ (long dashed curve), respectively.

in Fig. 8, indicates the existence of a transition from a decelerating phase to an accelerating one. It is important to point out that in the present model all relevant cosmological quantities (a , H , V) are singular at the beginning of the cosmological evolution corresponding to $t = 0$.

In the next Section, we shall present the cosmological solution for the particular case $c_1 = -2$.

4.2.2 Specific case $c_1 = -2$, $c_2 < 0$

For the case $c_1 = -2$, then the Lagrangian (77) takes the simple form

$$\mathcal{L} = -\lambda w^{2n} - m_0 \frac{\dot{w}}{w} \dot{z}, \quad (93)$$

where we have denoted the arbitrary constant m_0 as $m_0 = 2K\sqrt{-2nc_2}$. Again by inserting Eq. (93) into Euler-Lagrange Eqs. (41) and (42), we obtain

$$\frac{d}{dt} \left(\frac{\dot{w}}{w} \right) = 0, \quad (94)$$

$$\ddot{z} = \frac{2n\lambda}{m_0} w^{2n}. \quad (95)$$

respectively. By integrating Eqs. (94) and (95), we find

$$w(t) = h_2 e^{h_1 t}, \quad (96)$$

$$z(t) = z_1 e^{2nh_1 t} + h_3 t + h_4, \quad (97)$$

where h_i , $i = 1, 2, 3, 4$ are the arbitrary constants of integration, and we have denoted the arbitrary constant

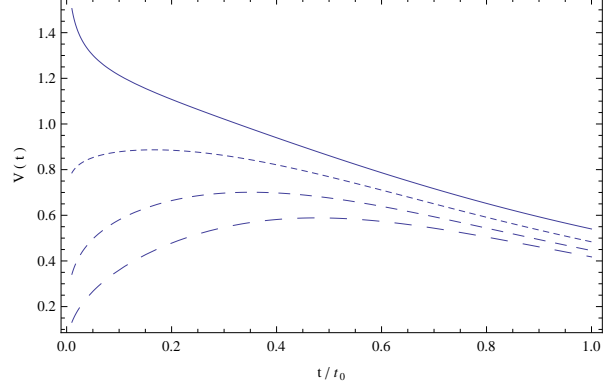


Fig. 7 Time variation of the scalar field potential $V(t)$ in the second Noether symmetry solution of the generalized STT cosmological model for different values of the parameter m : $m = 1.95$ (solid curve), $m = 2.05$ (dotted curve), $m = 2.20$ (dashed curve), and $m = 2.355$ (long dashed curve), respectively. The values of the integration constants have been fixed so that $\lambda\phi_0^{2(2m-3)/(m-2)} = 1$, while for n we have adopted the value $n = 2$.

z_1 as $z_1 = \frac{\lambda h_2^{2n}}{2nm_0 h_1^2}$. By substituting Eqs. (96) and (97) into Eq. (73), then we obtain the scale factor $a(t)$ given by

$$a(t) = A_0 (z_1 e^{2nh_1 t} + h_3 t + h_4)^{2/3} e^{-2nh_1 t/3}, \quad (98)$$

where $A_0 = \left[3h_2^{-n} \sqrt{-2c_2/n/2} \right]^{2/3}$. Without any loss of generality we can take for the constant h_4 the value zero, $h_4 = 0$. For the Hubble function we find

$$H(t) = \frac{2}{3} \left[\frac{h_3 (1 - 2h_1 n t)}{z_1 e^{2h_1 n t} + h_3 t} + h_1 n \right]. \quad (99)$$

At the initial moment $t = 0$ we have $a(0) = a_0 = A_0 z_1^{2/3}$, and $H(0) = H_0 = (2/3)(h_1 n + h_3/z_1)$, which gives $h_3/z_1 = 3H_0/2 - h_1 n$. Hence, by denoting $h_0 = nh_1$ and $\xi = 3H_0/2h_0$, we can write the scale factor and the Hubble function as

$$a(t) = a_0 [e^{2h_0 t} + (\xi - 1)h_0 t]^{2/3} e^{-2h_0 t/3}, \quad (100)$$

$$H(t) = \frac{2}{3} h_0 \left[\frac{(\xi - 1)(1 - 2h_0 t)}{e^{2h_0 t} + (\xi - 1)h_0 t} + 1 \right], \quad (101)$$

respectively. For the deceleration parameter $q(t)$, we obtain

$$q(t) = \frac{3(\xi - 1) [4e^{2h_0 t}(1 - h_0 t) + \xi - 1]}{2[(1 - \xi)h_0 t + e^{2h_0 t} + \xi - 1]^2} - 1. \quad (102)$$

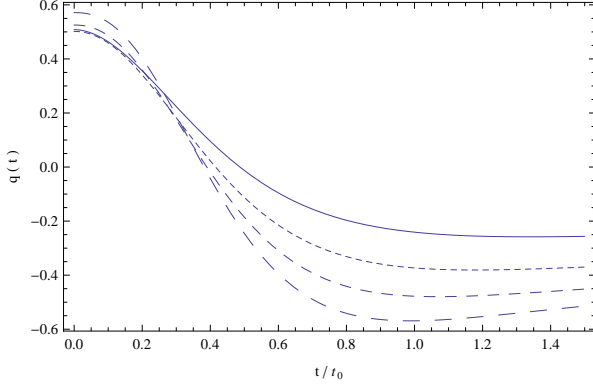


Fig. 8 Time variation of the deceleration parameter $q(t)$ in the second Noether symmetry solution of the generalized STT cosmological model for different values of the parameter m : $m = 1.95$ (solid curve), $m = 2.05$ (dotted curve), $m = 2.20$ (dashed curve), and $m = 2.355$ (long dashed curve), respectively.

By substituting Eqs. (96) and (97) into Eq. (74), then we obtain the scalar field

$$\phi(t) = h_2^2 e^{2h_1 t} \left[\frac{3}{2} \sqrt{\frac{-2c_2}{n}} (z_1 e^{2nh_1 t} + h_3 t + h_4) \right]^{-1/n}. \quad (103)$$

With the help of Eq. (103), since we have used $c_1 = -2$, the potential $V(t)$ can be easily obtained from the relation

$$V(\phi) = \lambda \phi^{3c_1 b_3} = \lambda \phi^{\frac{3nc_1}{1+2c_1}} = \lambda \phi^{2n}, \quad (104)$$

as

$$V(t) = V_1 e^{4h_0 t} [e^{2h_0 t} + (\xi - 1) h_0 t]^{-2}, \quad (105)$$

where $V_1 = -2n\lambda h_2^{4n}/9c_2 z_1^2$.

The time variations of the scale factor, of the Hubble function, of the scalar field potential and of the deceleration parameter for the third Noether symmetry cosmological solution are presented in Figs. 9-12 respectively.

Similarly to the behavior of the previous cosmological solutions, the scale factor, depicted in Fig 9 is a monotonically increasing function of time. The Hubble function, shown in Fig. 10, monotonically decreases during the expansion of the Universe, from its initial value H_0 . The scalar field potential, represented in Fig. 11, indicates a complex behavior of the scalar field. After a period of monotonic decrease, the field potential reaches a minimum value, and after that it starts increasing. The deceleration parameter, presented in Fig. 12, indicates that the evolution of the cosmological model starts in a decelerating phase, with $q > 0$.

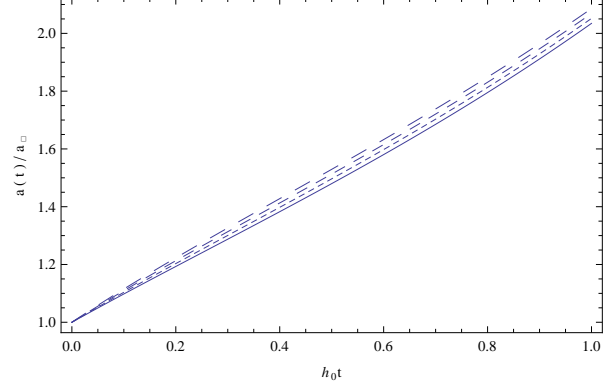


Fig. 9 Time variation of the scale factor $a(t)$ in the third Noether symmetry solution of the generalized STT cosmological model for different values of the parameter ξ : $\xi = 1.5$ (solid curve), $\xi = 1.6$ (dotted curve), $\xi = 1.7$ (dashed curve), and $\xi = 1.8$ (long dashed curve), respectively.

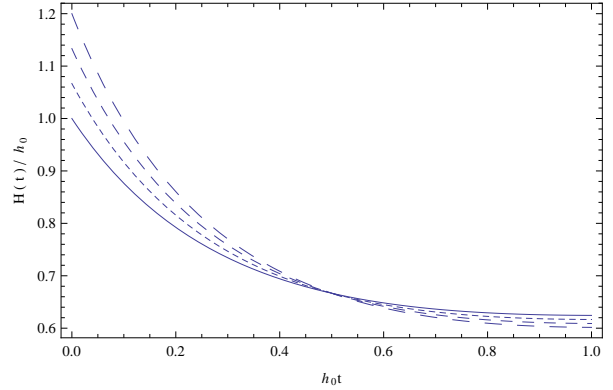


Fig. 10 Time variation of the Hubble function $H(t)$ in the third Noether symmetry solution of the generalized STT cosmological model for different values of the parameter ξ : $\xi = 1.5$ (solid curve), $\xi = 1.6$ (dotted curve), $\xi = 1.7$ (dashed curve), and $\xi = 1.8$ (long dashed curve), respectively.

Then the Universe enters into an accelerating phase, and in the large time limit the evolution becomes of de Sitter type, with $q \rightarrow -1$.

5 Conclusions and final remarks

In the present paper we have investigated, by using Noether symmetry techniques, the cosmological solutions of a generalized Brans-Dicke type STT gravitational model, in which there is a supplementary coupling between the scalar field and its kinetic energy term. From a mathematical point of view, we have obtained three analytical solutions satisfying the condition $L_{\mathbf{X}}\mathcal{L} = \mathbf{X}\mathcal{L} = 0$, where \mathcal{L} is the cosmological Lagrangian. Next by introducing two new variables z and

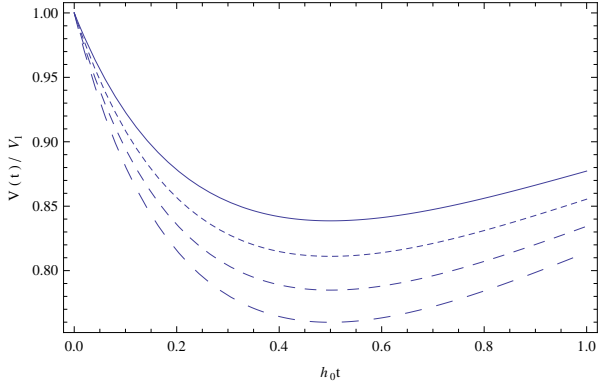


Fig. 11 Time variation of the scalar field potential $V(t)$ in the third Noether symmetry solution of the generalized STT cosmological model for different values of the parameter ξ : $\xi = 1.5$ (solid curve), $\xi = 1.6$ (dotted curve), $\xi = 1.7$ (dashed curve), and $\xi = 1.8$ (long dashed curve), respectively.

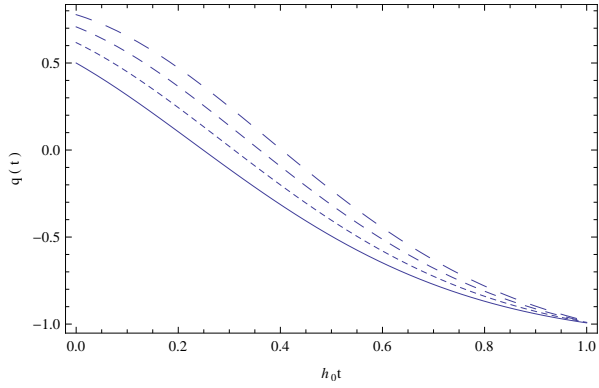


Fig. 12 Time variation of the deceleration parameter $q(t)$ in the third Noether symmetry solution of the generalized STT cosmological model for different values of the parameter ξ : $\xi = 1.5$ (solid curve), $\xi = 1.6$ (dotted curve), $\xi = 1.7$ (dashed curve), and $\xi = 1.8$ (long dashed curve), respectively.

w , defined by $z = z(a, \phi)$ and $w = w(a, \phi)$, and by fixing the numerical values of the arbitrary integration constants, we have obtained three cosmological solutions describing the dynamics of an FRW Universe. The cosmological properties of the solutions have been investigated in detail. The main, and common characteristic of these solutions is that they all describe the transition from decelerating to accelerating phases. Another interesting feature of the solutions is related to the complex time behavior of the scalar field potential, which for some numerical values of the model parameters can either increase to a maximum value, decreasing afterwards, as is the case in the second model (see Fig. 7), or decreases from a maximum to a minimum value, from which it starts to increase again (model three, Fig. 11). Of course, due to the presence of some ar-

bitrary numerical parameters the cosmological dynamics is strongly dependent on their numerical values. In the present approach we have fixed these unknown parameters from the initial conditions of the cosmological expansion, fixed at an arbitrary moment $t = 0$. But of course the exact numerical values of these parameters must be obtained from the full confrontation of the models with the observational results.

The Planck satellite has recently provided high-quality data of the CMB anisotropies (Adam et al. 2015; Ade et al. 2015a; Ade et al. 2015b). These data have open new perspectives for the study of the physical processes that dominated the cosmological evolution of the very early Universe. An important result that can be inferred from the observational data is that they support the inflationary paradigm (Guth 1981) as an explanation for the evolution of the primordial phase in the existence of the Universe. Hence, from observational point of view it turns out that presently the full set of cosmological observations can be explained by adopting a minimal theoretical setup, in which inflation is driven by a single scalar field ϕ (the inflaton), minimally coupled to gravity. The scalar field has a canonical kinetic term $\dot{\phi}^2/2$, and, moreover, its time evolution is determined by a flat potential $V(\phi)$ in the slow roll phase. Since in the time of inflation the energy of the physical processes is very high, there is no firm knowledge of the physics of particles and fields during inflation. That's why a large variety of scalar field potentials have been proposed so far in the literature, and the determination of the possible forms of $V(\phi)$ is a central issue of research in present day cosmology and theoretical physics. On the other hand the possibility of non-minimal couplings between the inflation field and gravity, like, for example, those introduced in Eq. (1) cannot be rejected a priori. In this context, the imposition of the Noether symmetry, which could be interpreted as a fundamental symmetry of nature, can provide a powerful method for the determination of the inflaton field potential. We have obtained the explicit form of several power-law type potentials, which could be relevant for the study of the inflationary epoch. In order to evaluate the cosmologically relevant parameters during inflation, the time evolution of the scalar field can be expressed in terms of the e-fold number N , given by $N \equiv \ln(a_e/a)$, where a_e is the value of the scale factor at the end of the inflation. N is related to the scalar field potential by the relation $N = (1/M_P^2) \int_{\phi_e}^{\phi} [V(\phi)/V'(\phi)] d\phi$ (Liddle & Leach 2003). Hence, if an inflationary potential is given, one can find first find $\phi(N)$. Once $\phi(N)$ is known, one can immediately obtain another important cosmological parameters, like the scalar spectral index, and the tensor to-scalar ratio, respectively. Therefore,

once these parameters have been computed, one can compare the predictions of the inflationary potentials obtained via Noether symmetry with the Planck 2015 results. Such a comparison may also fix the numerical values of the parameters (constants of integration) of the Noether potentials. Hence for all inflationary type Noether models non trivial constraints on the parameters can be obtained from the CMB data.

To conclude, in the present paper we have presented some theoretical tools that could help in the mathematical analysis of the scalar-tensor theories, and in the investigation of their cosmological implications.

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